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Almost Chebyshev Properties for L^1-Approximation of Continuous Functions

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1. INTRODUCTION

Let K be a compact subset of $\mathscr{R}^s$ $(s \geqslant 1)$ satisfying $K = \overline{\operatorname{Int}(K)}$ and denote by $C(K)$ the space of continuous, real-valued functions on K. Let $W_\infty = \{w \in L^\infty(K) : w > 0 \text{ on } K\}$ and, for $w \in W_\infty$, let $C_w(K)$ denote the space $C(K)$ endowed with the w-weighted L^1-norm

$$\|f\|_w = \int_K |f| \, w \, d\mu,$$

where μ denotes Lebesgue measure. If U is a finite dimensional subspace of $C(K)$ and $f \in C(K)$, let $P_w(f)$ denote the set of all best $\|\cdot\|_w$-approximations to f from U, and let $\mathscr{U}_w(U)$ denote the set of all functions in $C(K)$ that have unique best $\|\cdot\|_w$-approximations from U. We say that U is *Chebyshev* in $C_w(K)$ if $\mathscr{U}_w(U) = C(K)$.

Recently there has been considerable interest in characterizing the Chebyshev subspaces of $C_1(K)$ owing to the recent discovery that the spaces of spline functions on $[0, 1]$ with fixed knots are Chebyshev in $C_1[0, 1]$ (see [3, 21 and references therein] as well as the older result of Krein [19, p. 236]) that spaces satisfying the Haar condition on $(0, 1)$ are Chebyshev in $C_1[0, 1]$. The only complete characterizations of the Chebyshev subspaces of $C_1[0, 1]$ involve references to the best approximation problem or characterizations of best approximations rather than a structural property of the subspace [2, 22]. However, a structural property, called the A-property, that is satisfied by the spaces mentioned above was introduced by Strauss [23] and shown to be sufficient for a subspace to be Chebyshev in $C_1(K)$. A subspace U of $C(K)$ is said to satisfy the *A-property* (or be called an *A-space*) if for every $u \in U \backslash \{0\}$ and every continuous function $\sigma: K \backslash Z(u) \to \{-1, 1\}$ there exists $v \in U \backslash \{0\}$ such that

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$v=0$ a.e. on $Z(u)$ and $\sigma v \geqslant 0$ on supp(u). Here $Z(f)=\{x \in K: f(x)=0\}$ and supp$(f)=K \backslash Z(f)$ for $f \in C(K)$. This particular version of the A-property was given in [8]. Subsequently, Kroó [7] and Pinkus [13] showed that the A-property is necessary and sufficient for a finite dimensional subspace to be Chebyshev in $C_w(K)$ for all $w \in W_\infty$. In fact, necessity requires far fewer weight functions [9, 16]. When $s=1$ and $K=[0,1]$, Pinkus [13] gave a complete "spline-like" description of the A-spaces. In the multivariate setting, the situation is not so neat. For instance, when $s=2$ and K is a rectangle, the known A-spaces include those that reduce to univariate A-spaces (through multiplication by a positive continuous function or a diffeomorphic transformation of the domain) [10], the space of linear functions, and certain spaces of linear splines with triangular elements (see Sommer [20]). Unfortunately, most spaces of polynomials and tensor products of univariate spline spaces fail to satisfy the A-property. For the fixed weight function $w=1$, few examples of Chebyshev spaces are known (see [6, 20]). Particularly, it is not known whether the polynomials of total degree $\leqslant n$ $(n>2)$ or the tensor product of the univariate polynomial spaces of degree $\leqslant n, m$ $(n, m \geqslant 2)$ are Chebyshev in $C_1(K)$ when K is a rectangle in $\mathscr{R}^2$.

Since many of the important approximating spaces in the multivariate setting fail to be A-spaces, we weaken the requirement that $\mathscr{U}_w(U)=C(K)$ and examine the question of whether $\mathscr{U}_w(U)$ is dense in $C(K)$ relative to the $\|\cdot\|_1$-topology or the uniform norm $\|\cdot\|_\infty$-topology. In Section 2, we demonstrate that $\mathscr{U}_w(U)$ is $\|\cdot\|_w$-dense in $C(K)$ for any finite dimensional subspace U of $C(K)$. Of interest here is a recent result of Pinkus [14] that the metric projection P_1 admits a $\|\cdot\|_1$-continuous selection if and only if U is Chebyshev in $C_1(K)$. This result will follow immediately from our theorem in Section 2. In topologically assessing the prevalence of uniqueness for L^1-approximation in $C(K)$, it may be more pertinent to use the uniform norm. One reason for this is that $C(K)$ with the uniform norm is complete and in Section 3 we will show that if $\mathscr{U}_w(U)$ is $\|\cdot\|_\infty$-dense in $C(K)$, then $C(K) \backslash \mathscr{U}_w(U)$ is of first category in $C(K)$ relative to the $\|\cdot\|_\infty$-topology. A second reason is that computational errors tend to be uniformly small. Thus when $\mathscr{U}_w(U)$ is $\|\cdot\|_\infty$-dense in $C(K)$, if we "randomly" but continuously perturb f, then we are almost sure that the result has a unique best $\|\cdot\|_w$-approximation from U. Furthermore, in Section 3, when K is locally connected and the nontrivial elements of U have sparse zero sets, we give a necessary and sufficient condition for $\mathscr{U}_w(U)$ to be $\|\cdot\|_\infty$-dense in $C(K)$. Essentially the condition is that no continuous sign function annihilates U. In Section 4, we vary the weight functions and completely characterize those finite dimensional subspaces U of $C(K)$ for which $\mathscr{U}_w(U)$ is $\|\cdot\|_\infty$-dense in $C(K)$ for all $w \in W_\infty$. There is a striking similarity between the A-property and our condition—specifically, $Z(u)$ is replaced by Int $Z(u)$. We shall also see that the conditions of our theorems are easy to check as we apply them to multivariate polynomial spaces and tensor product spline spaces. We remark that topological assessment of the extent of uniqueness is not new. In [4], Garkavi defined a subspace U of a Banach space B to be *almost Chebyshev* in B if the set of elements of B not having unique best approximations from U is of first category in B. One of his primary tools was a lemma similar to our Lemma 3 which reduced the almost Chebyshev property to the density of the uniqueness set in B when B is separable and U is reflexive. In [5], Garkavi characterized the finite dimensional almost Chebyshev subspaces of $C(Q)$ (uniform approximation), and in [15] Rozema characterized the almost Chebyshev subspaces of $L^1(\Omega, \Sigma, \nu)$ and related this property to the nonexistence of continuous selections of the metric projection as we have done in the present context. In simplified form, if Q contains no isolated points, then the almost Chebyshev subspaces of $C(Q)$ are those for which all nontrivial elements have sparse zero sets, and if (Ω, Σ, ν) contains no atoms, then all finite dimensional subspaces are almost Chebyshev in $L^1(\Omega, \Sigma, \nu)$ (although none are Chebyshev). We refrain from using the term "almost Chebyshev" in the case of $\|\cdot\|_w$-density of $\mathscr{U}_w(U)$ since $C_w(K)$ is not complete. When $C(K)\backslash\mathscr{U}_w(U)$ is of first category in $C(K)$ with respect to the $\|\cdot\|_\infty$-topology, we call U *uniformly almost Chebyshev in* $C_w(K)$.

2. Density of Uniqueness in $C_w(K)$

In this section we demonstrate the density of $\mathscr{U}_w(U)$ in $C_w(K)$ for any finite dimensional subspace U of $C(K)$ and any $w \in W_\infty$. In fact, our result is stronger in that for every $f \in C(K)$ and $u_0 \in P_w(f)$, w_0 is a strongly unique best $\|\cdot\|_w$-approximation to continuous functions arbitrarily near f in the $\|\cdot\|_w$-sense. We say that u_0 is a *strongly unique* best $\|\cdot\|_w$-approximation to f from U if there is a positive constant $\gamma > 0$ such that

$$\|f-u\|_w \geqslant \|f-u_0\|_w + \gamma \|u-u_0\|_w$$

for all $u \in U$.

THEOREM 1. *Let U be a finite dimensional subspace of $C(K)$, $w \in W_\infty$, $f \in C(K)$ and $u_0 \in P_w(f)$. For any $\varepsilon > 0$ there exists $g \in C(K)$ such that $\|f-g\|_w < \varepsilon$ and u_0 is a strongly unique best $\|\cdot\|_w$-approximation to g from U.*

Before proving Theorem 1, we state two lemmas. The first is a well-known characterization of best L^1-approximations [18, 19] and the second is a characterization of strongly unique best L^1-approximations due to

Nürnberger [11]. Although both lemmas hold in general L^1-spaces, we only state them in the pertinent context.

LEMMA 1. *Let U be a finite dimensional subspace of $C(K)$, $w \in W_\infty$, $f \in C(K)\backslash U$, and $u_0 \in U$. The following are equivalent:*

(1) $u_0 \in P_w(f)$

(2) *for all* $u \in U$

$$\int_K u(\operatorname{sgn}(f-u_0))w\,d\mu \leqslant \int_{Z(f-u_0)} |u|\,w\,d\mu.$$

(3) *There exists measurable $\sigma: K \to \{-1, 1\}$ such that $\sigma = \operatorname{sgn}(f-u_0)$ on $\operatorname{supp}(f-u_0)$ and for all $u \in U$,*

$$\int_K \sigma u\,d\mu = 0. \tag{2.1}$$

In Lemma 1, generally we only have $|\sigma| \leqslant 1$ on $Z(f-u_0)$, but when the underlying measure space contains no atoms (as in the present context) we can assume that $|\sigma| = 1$ on $Z(f-u_0)$ (see [12, Lemma 2]).

LEMMA 2. *Let U be a finite dimensional subspace of $C(K)$, $w \in W_\infty$, and $f \in C(K)\backslash U$. Then 0 is a strongly unique best $\|\cdot\|_w$-approximation to f from U if and only if*

$$\int_{\operatorname{supp}(f)} u(\operatorname{sgn} f)w\,d\mu < \int_{Z(f)} |u|\,w\,d\mu$$

for all $u \in U\backslash\{0\}$.

Proof of Theorem 1. We write the proof only for $w = 1$. For arbitrary w, the same proof holds with μ replaced by the measure μ_w given by $d\mu_w = w\,d\mu$. Without loss of generality, we assume that $u_0 = 0$. By Lemma 1, there exists measurable $\sigma: K \to \{-1, 1\}$ so that $\sigma = \operatorname{sgn} f$ on $\operatorname{supp}(f)$ and (2.1) for all $u \in U$.

Fix $u \in S_U := \{u \in U: \|u\|_\infty = 1\}$. By (2.1), the sets $P_u := \{x \in K: \sigma(x)\,u(x) > 0\}$ and $N_u := \{x \in K: \sigma(x)\,u(x) < 0\}$ have positive measure. If $P_u \cap Z(f)$ has positive measure, choose $r_u > 0$ so that the set $A_u := \{x \in Z(f): \sigma(x)\,u(x) > r_u\}$ has positive measure. If $P_u \cap Z(f)$ has measure zero, then $P_u \cap \operatorname{supp}(f)$ has positive measure and thus is nonempty. Choose $x \in \operatorname{supp}(f)$ so that $\sigma(x)\,u(x) > 0$. Since σ is constant in a neighborhood of x, we can choose $r_u > 0$ and open A_u so that $x \in A_u \subseteq \operatorname{supp}(f)$ and $\sigma u > r_u$ on A_u. In either case, $\mathscr{A}_u := \{v \in S_U: \sigma v > 0$ on $A_u\}$ is a $\|\cdot\|_\infty$-neighborhood of u in S_U.

Since S_U is compact, there exist finitely many $u_1, ..., u_n \in S_U$ such that $S_U \subseteq \bigcup_{i=1}^n \mathscr{A}_{u_i}$. In particular, for every $u \in S_U$, $\sigma u > 0$ on A_{u_i}, for some $i = 1, ..., n$. Reorder the u_i's, if necessary, so that $A_{u_1}, ..., A_{u_k} \subseteq \operatorname{supp}(f)$ and $A_{u_{k+1}}, ..., A_{u_n} \subseteq Z(f)$. Since f is integrable, we can choose $\delta > 0$ so that $\int_T |f| \, d\mu < \varepsilon$ for any measurable subset T of K with $\mu(T) < \delta$. For $i = 1, ..., k$, choose nonempty open O_i and V_i so that $O_i \subseteq \bar{O}_i \subseteq V_i \subseteq A_{u_i}$ and $\mu(V_i) < \delta/(2k)$, and let $O' = \bigcup_{i=1}^k O_i$ and $V' = \bigcup_{i=1}^k V_i$. Thus O' and V' are nonempty open sets, $\mu(V') < \delta/2$, $O' \subseteq \bar{O}' \subseteq V' \subseteq V' \subseteq \operatorname{supp}(f)$, and for every $u \in S_U$, $\sigma u > 0$ on a subset of $Z(f) \cup O'$ of positive measure. Similarly, we can find nonempty open sets O'' and V'' so that $\mu(V'') < \delta/2$, $O'' \subseteq \bar{O}'' \subseteq V'' \subseteq \operatorname{supp}(f)$, and for each $u \in S_U$, $\sigma u < 0$ on a subset of $Z(f) \cup O''$ of positive measure. Letting $O = O' \cup O''$ and $V = V' \cup V''$, we see that O and V are nonempty open sets,

$$O \subseteq \bar{O} \subseteq V \subseteq \operatorname{supp}(f).$$

$$\mu(V) < \delta. \tag{2.2}$$

and for every $u \in S_U$

$$\mu\{x \in O \cup Z(f) : \sigma u > 0\} > 0 \tag{2.3}$$

and

$$\mu\{x \in O \cup Z(f) : \sigma u < 0\} > 0. \tag{2.4}$$

Now we choose $g \in C(K)$ satisfying $g = f$ on $K \backslash V$, $g = 0$ on $\bar{O}$, and $\operatorname{sgn} g = \operatorname{sgn} f$ and $|g| < |f|$ on $V \backslash \bar{O}$. Such a construction can be made using a Urysohn function ϕ where $\phi = 1$ on $K \backslash V$, $\phi = 0$ on $\bar{O}$, and $0 < \phi < 1$ on $V \backslash \bar{O}$ and letting $g = f\phi$. By (2.2),

$$\|f - g\|_1 = \int_V |f - g| \, d\mu = \int_V (|f| - |g|) \, d\mu \leqslant \int_V |f| \, d\mu < \varepsilon.$$

Also, $Z(g) = \bar{O} \cup Z(f)$ and $\operatorname{sgn} g = \operatorname{sgn} f = \sigma$ on $\operatorname{supp}(g) = \operatorname{supp}(f) \backslash \bar{O}$. By (2.1), (2.3), and (2.4), for $u \in S_U$,

$$\int_{\operatorname{supp}(g)} u(\operatorname{sgn} g) \, d\mu = \int_{\operatorname{supp}(g)} u\sigma \, d\mu = -\int_{Z(G)} u\sigma \, d\mu < \int_{Z(g)} |u| \, d\mu. \tag{2.5}$$

By homogeneity, (2.5) holds for all $u \in U \backslash \{0\}$, and by Lemma 2, 0 is a strongly unique best $\|\cdot\|_1$-approximation to g from U.

The density result now follows immediately.

Corollary 1. *For any finite dimensional subspace U of $C(K)$, $\mathscr{U}_w(U)$ is $\|\cdot\|_w$-dense in $C(K)$ for all $w \in W_\infty$.*

Pinkus' result on continuous selections also follows.

COROLLARY 2. *Let U be a finite dimensional subspace of $C(K)$ and $w \in W_\infty$. A necessary and sufficient conditions for P_w to admit a $\|\cdot\|_w$-continuous selection is that U be Chebyshev in $C_w(K)$.*

Proof. As is noted in Pinkus [14], sufficiency is well known. For necessity, suppose U is not Chebyshev in $C_w(K)$. Choose $f \in C_w(K)$ for which $P_w(f)$ contains two distinct functions u_0 and u_1. By Theorem 1, there exist sequences (g_k) and (h_k) in $C(K)$ where $\|g_k - f\|_w \to 0$, $\|h_k - f\|_w \to 0$, $P_w(g_k) = \{u_0\}$, and $P_w(h_k) = \{u_1\}$. For any selection G of P_w, $G(g_k) = u_0$ and $G(h_k) = u_1$. It is now clear that G cannot be $\|\cdot\|_w$-continuous at f.

3. UNIFORM DENSITY OF UNIQUENESS

When K is locally connected and $Z(u)$ has an empty interior for all non-trivial u in a finite dimensional subspace U of $C(K)$, we give necessary and sufficient conditions for U to be uniformly almost Chebyshev in $C_w(K)$ for fixed $w \in W_\infty$.

THEOREM 2. *Let K be locally connected, $w \in W_\infty$ and let U be a finite dimensional subspace of $C(K)$ for which $\operatorname{Int} Z(u) = \varnothing$ for all $u \in U \backslash \{0\}$. Then U is uniformly almost Chebushev in $C_w(K)$ if and only if there does not exist a continuous sign function $\sigma: K \to \{-1, 1\}$ such that*

$$\int_K \sigma u w \, d\mu = 0 \tag{3.1}$$

for all $u \in U$.

As in Section 2, we only prove Theorem 2 for $w = 1$ and note that replacing μ by μ_w yields the proof for any $w \in W_\infty$. We require three lemmas. The first demonstrates that in order to prove that U is uniformly almost Chebyshev in $C_1(K)$ it suffices to show that $\mathscr{U}_1(U)$ is $\|\cdot\|_\infty$-dense in $C(K)$. The statement and proof are similar to a result of Garkavi [4, p. 171]; we include the proof for completeness.

LEMMA 3. *Let U be a finite dimensional subspace of $C(K)$. If $\mathscr{U}_1(U)$ is $\|\cdot\|_\infty$-dense in $C(K)$, then U is uniformly almost Chebyshev in $C_1(K)$.*

Proof. For $k = 1, 2, \ldots$, let

$$\mathscr{F}_k = \left\{ f \in C(K) : \operatorname{diam}(P_1(f)) \geqslant \frac{1}{k} \right\},$$

where $\operatorname{diam}(A)$ denotes the $\|\cdot\|_\infty$-diameter of a subset A of $C(K)$. Evidently, $C(K)\backslash\mathscr{U}_1(U)=\bigcup_{k=1}^{\infty}\mathscr{F}_k$.

To show that each $\mathscr{F}_k$ is $\|\cdot\|_\infty$-closed, let (f_n) be a sequence in $\mathscr{F}_k$ and $f\in C(K)$ where $\|f_n-f\|_\infty\to 0$. For $n=1, 2, ...$, choose u_n, $v_n\in P_1(f_n)$ so that

$$\|u_n-v_n\|_\infty\geqslant\frac{1}{k}-\frac{1}{n}.$$

Now $\|u_n\|_1$, $\|v_n\|_1\leqslant 2\|f_n\|_1\to 2\|f\|_1$ since $\|\cdot\|_\infty$-convergence implies $\|\cdot\|_1$-convergence. Since $\dim U<\infty$, we may extract subsequences and assume that $u_n\to u$ and $v_n\to v$ where $u, v\in U$. (Since $\dim U<\infty$, these latter convergences are with respect to any norm on U.) Since $\|f_n-f\|_1\to 0$ and set valued metric projections onto finite dimensional spaces are upper semicontinuous, $u, v\in P_1(f)$. Further

$$\begin{aligned}\|u-v\|_\infty &\geqslant \|u_n-v_n\|_\infty-\|u-u_n\|_\infty-\|v-v_n\|_\infty\\ &\geqslant\frac{1}{k}-\frac{1}{n}-\|u-u_n\|_\infty-\|v-v_n\|_\infty.\end{aligned}$$

Letting $n\to\infty$, we see that $\operatorname{diam}(P_1(f))\geqslant\|u-v\|_\infty\geqslant 1/k$ so that $f\in\mathscr{F}_k$. Thus $\mathscr{F}_k$ is $\|\cdot\|_\infty$-closed.

Since $\mathscr{U}_1(u)$ is $\|\cdot\|_\infty$-dense in $C(K)$, each $\mathscr{F}_k$ has an empty $\|\cdot\|_\infty$-interior, and thus $C(K)\backslash\mathscr{U}_1(U)$ is of first category in $C(K)$ with respect to the $\|\cdot\|_\infty$-topology.

Our next lemma sheds light on the nature of nonuniqueness in best $\|\cdot\|_1$-approximation, and various versions of it have been used fruitfully in the literature. For $f\in C(K)$ and $u\in P_1(f)$, we call u an *interior best* $\|\cdot\|_1$-*approximation* to f from U if $Z(f-u)\subseteq Z(v-u)$ for all $v\in P_1(f)$. Specifically, an interior best approximation to f interpolates f on a minimal set for all best approximations.

LEMMA 4. *Let U be a finite dimensional subspace of $C(K)$. Then every $f\in C(K)$ has an interior best $\|\cdot\|_1$-approximation from U.*

Proof. If $P_1(f)$ is singleton, the statement is clear. Since $P_1(f)$ is convex and has finite dimension, it has a nonempty interior relative to its affine span. By translation, if necessary, suppose 0 is in the interior of $P_1(f)$ relative to its affine span. Let $v\in P_1(f)$. Then there is an $\alpha>0$ so that $\pm\alpha v\in P_1(f)$. Thus $\|f\|_1=\|f-\alpha v\|_1=\|f+\alpha v\|_1$ and since $f=\frac{1}{2}(f-\alpha v)+\frac{1}{2}(f+\alpha v)$, we have

$$\int_K\left|\frac{1}{2}(f-\alpha v)+\frac{1}{2}(f+\alpha v)\right|d\mu=\int_K\left(\frac{1}{2}|f-\alpha v|+\frac{1}{2}|f+\alpha v|\right)d\mu.$$

By the triangle inequality for absolute value, equality above, and continuity of f and v, $|\frac{1}{2}(f-\alpha v)+\frac{1}{2}(f+\alpha v)|=\frac{1}{2}|f-\alpha v|+\frac{1}{2}|f+\alpha v|$ on K. Hence, $(f-\alpha v)(f+\alpha v)\geqslant 0$ on K. Thus $f^2\geqslant \alpha^2 v^2$ on K and $Z(f)\subseteq Z(v)$.

The main content of this section in the next lemma.

LEMMA 5. *Suppose that K is locally connected, U is a finite dimensional subspace of $C(K)$, $f\in C(K)$, and u is an interior best $\|\cdot\|_1$-approximation to f from U. If $Z(f-u)\nsubseteq \operatorname{Int} Z(v)$ for every $v\in U\backslash\{0\}$, then for every $\varepsilon>0$ there exists $g\in \mathscr{U}_1(U)$ such that $\|f-g\|_\infty<\varepsilon$.*

Proof. Again we assume with no loss of generality that $u=0$. We first observe that given $\varepsilon>0$ there is a $\delta>0$ so that if $g\in C(K)$ and $\|f-g\|_1\leqslant\delta$, then $|v(x)|\leqslant\varepsilon/3$ for all $v\in P_1(g)$ and $x\in Z(f)$. This follows since P_1 is $\|\cdot\|_1-\|\cdot\|_1$ upper semicontinuous [19, p. 386] and $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent on U. That is, there is a $\delta>0$ so that if $g\in C(K)$, $\|f-g\|_1<\delta$ and $v\in P_1(g)$, then there exists $u\in P_1(f)$ so that $\|u-v\|_\infty<\varepsilon/3$. But $u=0$ on $Z(f)$ so that $|v|<\varepsilon/3$ on $Z(f)$.

We also note that if $\|f-g\|_1\leqslant 1$ and $v\in P_1(g)$, then

$$\|v\|_\infty\leqslant K\|v\|_1\leqslant 2K\|g\|_1\leqslant 2K(1+\|f\|_1):=M, \tag{3.2}$$

where K is the equivalence constant for the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ on U. Now define $H\colon K\to[0,\infty)$ by

$$H(y)=\sup\{|u(y)| : u\in U,\ \|u\|_\infty\leqslant M,\ |u|\leqslant\varepsilon/3 \text{ on } Z(f)\}. \tag{3.3}$$

Since the set over which the sup is taken is $\|\cdot\|_\infty$-compact and therefore equicontinuous there is an open neighborhood O on $Z(f)$ for which $H<2\varepsilon/3$ on O. Further, refine O so that $|f|<\varepsilon/3$ on O.

Let $\{u_1, ..., u_n\}$ be a basis for U. By the hypothesis that $Z(f)\nsubseteq \operatorname{Int} Z(v)$ for any $v\in U\backslash\{0\}$, $u_1, ..., u_n$ are linearly independent on O. Thus we may choose n distinct points $y_1, ..., y_n$ in O over which $u_1, ..., u_n$ are linearly independent. Thus

$$\det(u_i(y_j))_{i,j=1}^n\neq 0.$$

By the continuity of the u_1's, the local connectedness of K, and the regularity of μ, there exist disjoint open connected neighborhoods $V_1, ..., V_n$ of $y_1, ..., y_n$, respectively, contained in O, so that

$$\det(u_i(t_j))_{i,j=1}^n\neq 0 \tag{3.4}$$

when $t_j\in V_j$ $(j=1, ..., n)$ and

$$\mu(V_j)<\min(1,\delta)/(n\varepsilon) \tag{3.5}$$

$(j=1, \ldots, n)$. Since $K=\overline{\mathrm{Int}(K)}$, K has no isolated points and we may choose $s_j \in V_j \setminus \{y_i\}$ $(j=1, \ldots, n)$.

We now select $g \in C(K)$ satisfying the following conditions:

$$g(y_j)=H(y_j) \quad \text{and} \quad g(s_j)=-H(s_j) \quad (j=1, \ldots, n) \tag{3.6}$$

$$g=f \quad \text{on} \quad K \Big\backslash \left(\bigcup_{j=1}^{n} V_j \right) \tag{3.7}$$

$$|g| \leqslant 2\varepsilon/3 \quad \text{on} \quad \bigcup_{j=1}^{n} V_j. \tag{3.8}$$

Since $|H| \leqslant 2\varepsilon/3$ and $|f| < \varepsilon/3$ on $\bigcup_{j=1}^{n} V_j \subseteq O$, g can be obtained. By (3.7) and (3.8), $\|f-g\|_\infty < \varepsilon$. Also, by (3.7) and (3.5),

$$\|f-g\|_1 = \sum_{j=1}^{n} \int_{V_j} |f-g| \, d\mu \leqslant \varepsilon \sum_{j=1}^{n} \mu(V_j) < \min(\delta, 1). \tag{3.9}$$

To see that $g \in \mathscr{U}_1(U)$, using Lemma 4, let u be an interior best $\|\cdot\|_1$-approximation to g from U. By (3.9), $\|u\|_\infty \leqslant M$ and $|u| \leqslant \varepsilon/3$ on $Z(f)$. By the definition of H and (3.6), $u(s_j) \geqslant g(s_j)$ and $u(y_j) \leqslant g(y_j)$ $(j=1, \ldots, n)$. Since each V_j is connected, $V_j \cap Z(g-u) \neq 0$ $(j=1, \ldots, n)$. Choose $t_j \in V_j \cap Z(g-u)$ $(j=1, \ldots, n)$. If $P_1(g)$ contained an element v other than u, then $\{t_1, \ldots, t_n\} \subseteq Z(g-u) \subseteq Z(v-u)$. Since $v-u \neq 0$, the basis $u_1, \ldots, u_n$ would be linearly dependent on $\{t_1, \ldots, t_n\}$ so that

$$\det(u_i(t_j))_{i,j=1}^{n} = 0$$

contrary to (3.4). Thus $u \in \mathscr{U}_1(U)$.

Proof of Theorem 2. Suppose that a continuous $\sigma: K \to \{-1, 1\}$ exists satisfying (3.1) for all $u \in U$. For all $f \in C(K)$ with $\|f-\sigma\|_\infty < \frac{1}{2}$ and $v \in U$ with $\|v\|_\infty < \frac{1}{2}$, $\sigma = \mathrm{sgn}(f-v)$. By (3.1) and Lemma 1, $v \in P_1(f)$. Thus $\mathscr{U}_1(U)$ is not $\|\cdot\|_\infty$-dense in $C(K)$ and U is not uniformly almost Chebyshev in $C_1(K)$. Conversely, suppose that no continuous sign function annihilates U. By Lemmas 4 and 5 and the condition that $\mathrm{Int}\, Z(u) = \varnothing$ for all $u \in U \setminus \{0\}$, it suffices to show that $Z(f-v) \neq \varnothing$ for all $f \in C(K)$ and $v \in P_1(f)$. If $Z(f-v) = \varnothing$ for some $f \in C(K)$ and $v \in P_1(f)$, then $\sigma = \mathrm{sgn}(f-v)$ would be a continuous sign function and by Lemma 1, (3.1) would hold for all $u \in U$, a contradiction.

EXAMPLE 1. *Let* $K=[a, b] \times [c, d] \subseteq \mathscr{R}^2$. Since K is connected, the only continuous sign functions on K are constant. Further if $U=\mathrm{span}\{p_1(x, y), \ldots, p_n(x, y)\}$ where $p_1, \ldots, p_n$ are polynomials, then $Z(p)$ has empty interior for all nontrivial elements p of U. By Theorem 2, U can fail to be uniformly almost Chebyshev in $C_1(K)$ precisely when $\int_K p_i\,d\mu = 0$ for $i = 1, ..., n$. If $p_1 = 1$, then it is easy to see that U is uniformly almost Chebyshev in $C_w(K)$ for all $w \in W_\infty$.

EXAMPLE 2. We see that the condition $\operatorname{Int} Z(u) = \varnothing$ for all $u \in U \backslash \{0\}$ cannot be removed from Theorem 2. Let $K = [-3, 3]$ and $U = sp\{u_1, u_2\}$ where $u_1 \equiv 1$ and

$$u_2(x) = \begin{cases} -(x+2) & \text{if } -3 \leqslant x < -2 \\ 0 & \text{if } -2 \leqslant x \leqslant 2 \\ x-2 & \text{if } 2 < x \leqslant 3. \end{cases}$$

No constant sign function annihilates U, but U is not uniformly almost Chebyshev in $C_1[-3, 3]$. We refer the reader to the proof of necessity in Theorem 3 for the latter assertion.

4. VARYING WEIGHT FUNCTIONS

For a fixed weight function $w \in W_\infty$, Theorem 2 characterizes those subspaces U that are uniformly almost Chebyshev in $C_w(K)$ under the condition that $Z(u)$ has an empty interior for all $u \in U \backslash \{0\}$. In this section we circumvent this condition by letting w vary.

THEOREM 3. *Let K be locally connected and U be a finite dimensional subspace of $C(K)$. Then U is uniformly almost Chebyshev in $C_w(K)$ for all $w \in W_\infty$ if and only if for every $u \in U \backslash \{0\}$ and continuous $\sigma: K \backslash \operatorname{Int} Z(u) \to \{1, -1\}$ there exists $v \in U \backslash \{0\}$ such that $v = 0$ on* $\operatorname{Int} Z(u)$ *and $\sigma v \geqslant 0$ on* $K \backslash \operatorname{Int} Z(v)$.

We note the striking resemblance between Theorem 3 and the A-property. The condition of Theorem 3 can be much easier to check since we need to check fewer sign functions in this case. This will be born out with an example on tensor product spline functions.

Necessity of the condition requires a lemma on moments which was used by Kroó [8] and Schmidt [16]. We state the version used in [16].

LEMMA 6. *Let (Ω, Σ, ν) be a positive, finite measure space, $S =$ span$\{s_1, ..., s_n\}$ be an n-dimensional subspace of $L^\infty(\Omega)$, and W be a convex cone in $L^\infty(\Omega)$ satisfying*

$$\text{if } q \in L^\infty(\Omega) \text{ and } \int_\Omega qw\,d\nu \geqslant 0 \text{ for all } w \in W, \text{ then } q \geqslant 0\ \nu \text{ a.e. on } \Omega \tag{4.1}$$

and let

$$A_n = \left\{ \left(\int_\Omega w s_i \, dv \right)_{i=1}^n : w \in W \right\} \subseteq \mathscr{R}^n.$$

If S contains no nontrivial functions that are nonnegative v *a.e. on* Ω, *then* $A_n = \mathscr{R}^n$.

Proof of Theorem 3. For sufficiency, assume that the given condition holds and fix $w \in W_\infty$. By Lemma 5 (with μ replaced by μ_w) it suffices to show that for every $f \in C(K)$ and $u \in P_w(f)$, $Z(f-u) \nsubseteq \operatorname{Int} Z(v)$ for all $v \in U \setminus \{0\}$. Let $f \in C(K)$ and suppose that $0 \in P_w(f)$ without loss of generality. Assume that $Z(f) \subseteq \operatorname{Int} Z(v)$ for some $v \in U \setminus \{0\}$. Since $\operatorname{sgn} f$ is continuous on $K \setminus Z(f)$, it is continuous on $K \setminus \operatorname{Int} Z(v) \subseteq K \setminus Z(f)$. By hypothesis, there exists $u \in U \setminus \{0\}$ so that $u = 0$ on $\operatorname{Int} Z(v) \supseteq Z(f)$ and $u(\operatorname{sgn} f) \geqslant 0$ on $K \setminus Z(v)$. As a result,

$$\int_{Z(f)} |u| \, w \, d\mu = 0 < \int_{K \setminus Z(f)} u(\operatorname{sgn} f) w \, d\mu$$

which contradicts Lemma 1(2). Sufficiency is now proven.

For necessity, assume the condition fails. Then there exists an open set O in K and continuous $\sigma: K \setminus O \to \{-1, 1\}$ such that the subspace $U_1 = \{u_1 \in U: O \subseteq Z(u_1)\}$ is nontrivial and contains no nontrivial element u_1 such that $\sigma u_1 \geqslant 0$ on $K \setminus O$. Let U_2 be a complementary subspace of U_1 in U so that $U = U_1 \oplus U_2$.

We obtain $w \in W_\infty$ so that $\mathscr{U}_w(U)$ is not $\|\cdot\|_\infty$-dense in $C(K)$. We have that the subspace $\sigma U_1 = \{\sigma u_1 : u_1 \in U_1\}$ contains no nontrivial elements that are nonnegative on $K \setminus O$. Indeed, σU_1 contains no functions that are nonnegative a.e. on $K \setminus O$ because $u_1 = 0$ on $Bdy(K \setminus O)$ for all $u_1 \in U_1$. By Lemma 6, there exists a weight function w defined on $K \setminus O$ such that

$$\int_{K \setminus O} \sigma u_1 w \, d\mu = 0$$

for all $u_1 \in U_1$. (As is noted in [16], we can choose w to be positive and continuous on $K \setminus O$.)

If $U_2 = \{0\}$, the proof proceeds just as in the proof of necessity in Theorem 2. We suppose that U_2 is nontrivial. By definition of U_1, no nontrivial element of U_2 vanishes identically on O. Choosing $\dim U_2$ points in O on which a basis for U_2 is linearly independent and a closed neighborhood of this set of points contained in O, we have a closed set $F \subseteq O$ where

$$\int_F |u_2| \, d\mu > 0$$

for all $u_2 \in U_2 \backslash \{0\}$. We extend w to all of K so that

$$\int_{K \backslash O} |u_2| \, w \, d\mu + \int_{O \backslash F} |u_2| \, w \, d\mu < \int_F |u_2| \, w \, d\mu \tag{4.2}$$

for all $u_2 \in U_2 \backslash \{0\}$. We could choose $w \equiv 1$ on $O \backslash F$ and $w \equiv c$ on F where

$$c = 1 + \sup_{u_2 \in S_{U_2}} \left(\int_{K \backslash O} |u_2| \, w \, d\mu + \int_{O \backslash F} |u_2| \, w \, d\mu \right) \Big/ \int_F |u_2| \, d\mu$$

and S_{U_2} is the $\|\cdot\|_\infty$-unit ball of U_2. However, w could also be extended continuously.

We now define $f \in C(K)$ for which all continuous functions uniformly near f have nonunique best $\|\cdot\|_w$-approximations. Since σ is continuous on the closed set $K \backslash O$, we can choose f to be continuous on K satisfying

$$f = \sigma \qquad \text{on } K \backslash O,$$

$$f = 0 \qquad \text{on } F,$$

and

$$0 < |f| < 1 \qquad \text{on } O \backslash F.$$

By (4.2) and Lemma 2, 0 is the unique best $\|\cdot\|_w$-approximation to f from U_2. Since metric projections are continuous at points having unique best approximations and $\|\cdot\|_\infty$-convergence implies $\|\cdot\|_1$-convergence, we can find $0 < \delta < \frac{1}{3}$ so that if $g \in C(K)$ and $\|f - g\|_\infty < \delta$, then $\|u_2\|_\infty < \frac{1}{3}$ for every best $\|\cdot\|_w$-approximation to g from U_2.

We finally show that if $g \in C(K)$ and $\|f - g\|_\infty < \delta$, then g has nonunique best $\|\cdot\|_w$-approximations from $U = U_1 \oplus U_2$. Let $g \in C(K)$, $\|f - g\|_\infty < \delta$, u_2 be a best $\|\cdot\|_w$-approximation to g from U_2, and u_1 any element of U_1 with $\|u_1\| < \frac{1}{3}$. By choice of δ, g, and u_2,

$$|u_1 + u_2| < \tfrac{2}{3} < |g|$$

on $K \backslash O$. Hence $Z(g - (u_1 + u_2)) \subseteq O$ and $\operatorname{sgn}(g - (u_1 + u_2)) = \sigma$ on $K \backslash O$. Further, since $u_1 \equiv 0$ on O, $Z(g - (u_1 + u_2)) = Z(g - u_2)$. Now for any $v = v_1 + v_2 \in U$ where $v_1 \in U_1$ and $v_2 \in U_2$, we have using Lemma 1 that

$$\int_{\operatorname{supp}(g - (u_1 + u_2))} (v_1 + v_2) \operatorname{sgn}(g - (u_1 + u_2)) \, w \, d\mu$$

$$= \int_{K \backslash O} \sigma (v_1 + v_2) \, w \, d\mu + \int_{O \backslash Z(g - u_2)} (v_1 + v_2) \operatorname{sgn}(g - u_2) \, w \, d\mu$$

$$= \int_{K \setminus O} \sigma v_2 w \, d\mu + \int_{O \setminus Z(g-u_2)} v_2 \operatorname{sgn}(g-u_2) w \, d\mu$$

$$\leqslant \int_{Z(g-u_2)} |v_2| \, w \, d\mu$$

$$= \int_{Z(g-(u_1+u_2))} |v_1+v_2| \, w \, d\mu.$$

So by Lemma 1, $u_1+u_2 \in P_w(g)$ and g has nonunique best $\|\cdot\|_w$ approximations from U. The proof is now complete.

We conclude this paper by showing that the spaces of tensor product spline functions satisfy the condition of Theorem 3 but are not A-spaces. We shall refer to Schumaker [17] for all necessary properties of splines. Let $K=[a,b]\times[c,d]\subseteq \mathscr{R}^2$. Let m and n be integers greater than 1 and specify knot sequences $\Delta_1=\{a=x_0<x^1<\cdots<x_k<x_{k+1}=b\}$ and $\Delta_2=\{c=y_0<y_1<\cdots<y_l<y_{l+1}=d\}$ and corresponding multiplicity vectors $\mathscr{M}_1=(m_1,...,m_k)$ and $\mathscr{M}_2=(n_1,...,n_l)$ where $1\leqslant m_i\leqslant m-1$ $(i=1,...,k)$ and $1\leqslant n_j\leqslant n-1$ $(j=1,...,l)$. Define $\mathscr{S}_1=\mathscr{S}(\Pi_{m-1},\mathscr{M}_1,\Delta_1)$ to be the set of all functions s on $[a,b]$ such that the restriction of s to $[x_i,x_{i+1}]$ is a polynomial of degree $\leqslant m-1$ $(i=0,...,k)$ and s, $Ds,...,D^{m-1-m_i}s$ are continuous at x_i $(i=1,...,k)$. Here Π_p denotes the set of polynomials of degree p or less and D denotes the differentiation operator. The space $\mathscr{S}_1$ is called a space of polynomial spline functions with fixed knots and has dimension

$$M:=\dim \mathscr{S}_1=m+\sum_{i=1}^{k} m_i.$$

We define $\mathscr{S}_2=\mathscr{S}(\Pi_{n-1},\mathscr{M}_2,\Delta_2)$ analogously, and $\mathscr{S}_2$ has dimension

$$N:=\dim \mathscr{S}_2=n+\sum_{j=1}^{l} n_j.$$

Note that the multiplicity vectors are chosen so that $\mathscr{S}_1\subseteq C[a,b]$ and $\mathscr{S}_2\subseteq C[c,d]$. As in [17, p. 116], for suitable extended partitions of Δ_1 and Δ_2, we can construct normalized B-spline bases $\{B_1,...,B_M\}$ and $\{\bar{B}_1,...,\bar{B}_N\}$ for $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively. The tensor product $\mathscr{S}=\mathscr{S}_1\otimes\mathscr{S}_2$ of $\mathscr{S}_1$ and $\mathscr{S}_2$ is the subspace of $C(K)$ having basis $\{B_{ij}: i=1,...,M,$ $j=1,...,N\}$ where $B_{ij}(x,y)=B_i(x)\,\bar{B}_j(y)$.

Of interest to us are the supports of the B-splines. Specifically, supp B_i is an interval in $[a,b]$ which is open relative to $[a,b]$ and has endpoints in the knot sequence Δ_1. Further $B_i>0$ on supp B_i. A similar description of supp $\bar{B}_j$ holds, and supp $B_{ij} = (\text{supp } B_i) \times (\text{supp } \bar{B}_j)$. We shall call any rectangle $(x_i, x_{i+1}) \times (y_j, y_{j+1})$ a cell in K.

THEOREM 4. *The space $\mathscr{S}$ is uniformly almost Chebyshev in $C_w(K)$ for all $w \in W_\infty$.*

Proof. We consider any nonzero function

$$s = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} B_{ij}.$$

in $\mathscr{S}$. Since $s \neq 0$, some $c_{pq} \neq 0$. We show that if $c_{pq} \neq 0$, then supp $B_{pq} \subseteq K \backslash \text{Int } Z(s)$. In this case, any continuous sign function σ on $K \backslash \text{Int } Z(s)$ would be constant on the connected set supp B_{pq}. Hence, $B_{pq} = 0$ on Int $Z(s)$ and $\sigma(\pm B_{pq}) \geqslant 0$ on $K \backslash \text{Int } Z(s)$.

To prove our assertion suppose supp $B_{pq} \not\subseteq K \backslash \text{Int } Z(s)$. Then some cell $(x_\eta, x_{\eta+1}) \times (y_\zeta, y_{\zeta+1})$ contained in supp B_{pq} contains a point of Int $Z(s)$. Since s is polynomial in two variables over this cell, $s \equiv 0$ on $(x_\eta, x_{n+1}) \times (y_\zeta, y_{\zeta+1})$. Let $B_\alpha, ..., B_\beta$ be the B-splines for $\mathscr{S}_1$ whose supports contain $(x_\eta, x_{\eta+1})$ and $\bar{B}_\gamma, ..., \bar{B}_\delta$ be the B-splines for $\mathscr{S}_2$ whose supports contain $(y_\zeta, y_{\zeta+1})$. Then

$$\sum_{i=\alpha}^{\beta} \left(\sum_{j=\gamma}^{\delta} c_{ij} \bar{B}_j(y) \right) B_i(x) = 0$$

for all $x \in (x_\eta, x_{\eta+1})$ and $y \in (y_\zeta, y_{\zeta+1})$. By [17; p. 169, Theorem 4.65], $B_\alpha, ..., B_\beta$ are linearly independent over $(x_\eta, x_{\eta+1})$. Thus

$$\sum_{j=\gamma}^{\delta} c_{ij} \bar{B}_j(y) = 0$$

for all $y \in (y_\zeta, y_{\zeta+1})$ and $i = \alpha, ..., \beta$. As above, $\bar{B}_\gamma, ..., \bar{B}_\sigma$ are linearly independent on $(y_\zeta, y_{\zeta+1})$, and, hence, $c_{ij} = 0$ $(i = \alpha, ..., \beta, j = \gamma, ..., \delta)$. In particular, $c_{pq} = 0$ and the assertion is proven.

Finally, we remark that S is not a A-space in $C(K)$. We outline the proof only. Consider the first two B-splines B_1, B_2 and $\bar{B}_1$, $\bar{B}_2$ in $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively. By [17, Theorem 4.65], each pair forms a Haar System on the knot intervals (x_0, x_1) and (y_0, y_1) and their supports do not extend beyond x_2 and y_2. Construct $s_1 \in sp\{B_1, B_2\}$ and $s_2 \in sp\{\bar{B}_1, \bar{B}_2\}$ where s_1 has one zero α in (x_0, x_1) and s_2 has one zero γ in (y_0, y_1). Consider the nontrivial function s in $\mathscr{S}$ given by

$$s(x, y) = s_1(x)\, s_2(y).$$

Now $Z(s)$ includes all (x, y) where $x_2 \leqslant x \leqslant b$ or $y_2 \leqslant y \leqslant d$ or $x = \alpha$ or $y = \gamma$. Define $\sigma: K \backslash Z(s) \to \{-1, 1\}$ by

$$\sigma(x, y) = \begin{cases} -1 & \text{if } a \leqslant x < \alpha \text{ and } c \leqslant y < \gamma \\ 1 & \text{otherwise} \end{cases}$$

If $\mathscr{S}$ were an A-space, then $\mathscr{S}$ would contain a nontrivial $\bar{s}$ where $\bar{s} = 0$ a.e. on $Z(s)$ and $\sigma \bar{s} \geqslant 0$ on $K \backslash Z(s)$. Then $\bar{s} \equiv 0$ on every cell except possibly $(x_i, x_{i+1}) \times (y_j, y_{j+1})$ $(i, j = 0, 1)$. But since $\bar{s}$ is a polynomial in two variables on the cell $(x_0, x_1) \times (y_0, y_1)$, the sign changes in that cell force $\bar{s} \equiv 0$ there. So $\operatorname{supp}(\bar{s})$ must be contained in the closure of the union of the cells $(x_1, x_2) \times (y_0, y_1)$, $(x_0, x_1) \times (y_1, y_2)$, and $(x_1, x_2) \times (y_1, y_2)$. If (x_1, x_2) is not the support of any of the B-splines $B_1, ..., B_m$ and (y_1, y_2) is not the support of any of the B-splines $\bar{B}_1, ..., \bar{B}_N$, then the argument in the proof of Theorem 4 shows that $\bar{s} = 0$, a contradiction. We omit the remaining cases but note that they involve constructions similar to $\bar{s}$ in other cells.

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